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The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings

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ABSTRACT. Let X be a Banach space, K a nonempty closed convex subset of X and $T : K \to K$ a Lipschitzian strictly pseudo-contractive mapping. We show that, in order to approximate the fixed point of T, instead of the Mann iteration, usually considered by many authors, we may use a simpler method, i.e., the Krasnoselskij iterative process, for which, in addition, it is also possible to find the fastest iteration to compute the fixed point. Subsidiary, it is also pointed out that an assumption like $\lim_{n\to\infty} \alpha_n = 0$, involved in most convergence theorems for Mann iteration existing in literature, appears to be artificial and not necessary.

1. INTRODUCTION

Let X be a Banach space and $T : X \to X$ a mapping. It is known, by the classical Banach contraction mapping principle, that if T is a L-contraction, i.e., a mapping for which there exists $0 < L < 1$ such that

(1.1) $\|Tx - Ty\| \leq L\,\|x - y\|$ for all $x, y \in X$,

then T has a unique fixed point $p \in X$, which can be obtained by means of Picard iteration, that is, the sequence of successive approximations $\{x_n\}$ defined by

(1.2) $x_{n+1} = Tx_n,\ n \geq 0,$

converges to p, as $n \to \infty$, for any initial approximation $x_0 \in X$.

If in (1.1) we have $L \geq 1$, then the above result is no more valid. In the case T is nonexpansive, i.e. $L = 1$, X is uniformly convex, and K is a nonempty closed convex and bounded subset of X, then the Browder-Göhde-Kirk fixed point theorem (see [6], [15], [17]) still ensure the existence of a fixed point of $T : K \to K$. But, unlike in the case of the Banach contraction principle, trivial examples, see [1], for instance, show that the sequence of successive approximations (1.2) may fail to converge to the fixed point even in the case when T has a unique fixed point.

To remove these difficulties, Krasnoselskij [18] observed that the averaged mapping

$$T_{1/2} = \frac{1}{2}(I + T),\ I = \text{the identity map},$$

which is also nonexpansive, possesses the same fixed points as T, and has a better asymptotic behavior than T itself and, therefore, can be used as an iteration function to approximate fixed points of T. Briefly, he proved that if X and K are as

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above and T is nonexpansive and compact, then the successive approximations defined by the iteration mapping $T_{1/2}$, i.e. the sequence

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ n \geq 0, \tag{1.3}$$

converges strongly to a fixed point of T.

Schaefer [27] extended the previous result, by using the more general iteration

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0 \text{ and } \lambda \in [0,1], \tag{1.4}$$

usually called *Krasnoselskij iteration* and which corresponds to the averaged mapping

$$T_\lambda = (1-\lambda)I + \lambda T,\ \lambda \in [0,1]. \tag{1.5}$$

The most general iterative scheme now intensively studied for approximating fixed points of nonexpansive mappings is the *Mann iteration* $\{x_n\}$, see [22], defined by $x_0 \in K$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geq 0, \tag{1.6}$$

where $\{\alpha_n\} \in [0,1]$ is a sequence of real numbers satisfying appropriate conditions. One such kind of condition is

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \tag{1.7}$$

It is clear that if we take $\alpha_n \equiv \lambda(const.),\ \lambda \in (0,1]$, (1.7) is satisfied and so (1.6) reduces to the Krasnoselskij iteration (1.4), which in turn gives Picard iteration (1.2) for $\lambda \equiv 1$.

There are a lot of recent papers concerning the convergence of the Mann iteration, see [7]-[14], [16], [18]-[26], [28] and references therein, and especially [1], for a comprehensive bibliography, but the great majority of them are obtained by imposing the following condition on the sequence $\{\alpha_n\}$:

$$\lim_{n\to\infty} \alpha_n = 0. \tag{1.8}$$

As pointed out in [1, Chapter 9, Example 9.2] and also shown by Example 2, in most cases condition (1.8) is not necessary for the convergence of Mann iteration and appears to be an artificial assumption, being tributary to the technique of proof used by the authors.

In this context, we remark that unless the case of Mann iteration, in the case of Halpern's fixed point iterative method, conditions (1.8) and (i) in this paper are known to be *necessary* for the convergence of that method, see [11]. Remind that Halpern's iteration [29] is defined by $x_0 \in K$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n,\ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence of real numbers in $[0,1]$, $T : K \to K$ is a given self-operator and $u \in K$ is arbitrary.

Example 1.1. ([1]) Let $X = \mathbb{R}$ with the usual norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ be a function given by $Tx = \frac{1}{x}$, for all x in K. Then:

(a) T is Lipschitzian with constant $L = 4$;

(b) T is strictly pseudocontractive, i.e., for $t > 1$

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

which is equivalent to

$$|x - y| \leq |x - y| \cdot \left|1 + r + \frac{rt}{xy}\right|,$$

obviously true for all $x, y \in K$, and $r > 0$;

(c) $F(T) = \{1\}$, where $F(T) = \{x \in K|\ Tx = x\}$;

(d) The Picard iteration associated to T does not converge to the fixed point of T, for any $x_0 \in K \setminus \{1\}$;

(e) The Krasnoselskij iteration associated to T converges to 1 for any $x_0 \in K$ and $\lambda \in (0, 1/16)$;

(f) The Mann iteration associated to T with $\alpha_n = \dfrac{n}{2n+1}$, $n \geq 0$ and $x_0 = 2$ converges to 1, the unique fixed point of T : $x_1 = 2$; $x_2 = 1.5$; $x_3 = 1.166$, $x_4 = 1.034$; $x_5 = 1.0042$; $x_6 = 1.00397$; $x_7 = 1.000031$;
$x_8 = 1.000002$ and $x_9 = 1$; However, $\alpha_n \nearrow \dfrac{1}{2}$ as $n \to \infty$ and so condition (1.8) is not satisfied.

But, from a computational point of view, when two or more iterative methods are available in order to approximate fixed points of a mappings in a certain class, it is natural to choose the simpler method, when known, in order to avoid complicated computations. On the other hand, it is clear that Krasnoselkij iteration method (1.4) is computationally simpler than the Mann iteration procedure (1.6).

Starting from the fact that many papers published in the last decade are devoted to the approximation of fixed points of several classes of mappings that include nonexpansive mappings, in this paper we show that, in the case of Lipschitzian strictly pseudo-contractive operators, the Krasnoselskij iteration suffices to approximate fixed points. Moreover, we also show that amongst all Krasnoselskij iterations that converge to the fixed point of such operators, we may select the *fastest* iteration, in some sense. This is indeed a very important achievement in view of concrete applications of fixed point iteration procedures.

The results in this paper open a new important direction of investigation: to analyze all convergence theorems for Mann iteration, Mann-type iteration etc., based on condition (1.8), in order to decide whether or not this assumption is indeed necessary for the convergence of that iteration and, secondly, to investigate if Krasnoselskij iteration could really replace Mann iteration for these classes of operators.

2. Convergence theorems for Mann iteration

Let X be a normed linear space. A mapping $T : D(T) \subset X \to X$ is said to be *strongly* (or *strictly*) *pseudo-contractive* if there exists $t > 1$ such that

(2.9) $$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all $x, y \in D(T)$ and $r > 0$.

If $t = 1$, then T is said to be *pseudo-contractive*. A mapping T is strongly pseudocontractive if and only if $I - T$ is a strongly accretive mapping, which is equivalent to the fact that the inequality

$$\| x - y\| \leq \| x - y + r\,[\,(I - T - kI)\,x - (I - T - kI)\,y\,]\| \tag{2.10}$$

holds for any $x, y \in K$ and any $r > 0$ (where $k = \dfrac{t-1}{t}$).

In the following, the set of fixed points of T will be denoted by $F(T)$; the Lipschitzian constant of T will be denoted by $L(\geq 1)$, while by k we denote the constant of strictly pseudo-contractiveness of T.

The main result in Liu [20] is the next theorem.

Theorem 2.1. *Let X be a Banach space and K a nonempty closed convex and bounded subset of X. Let $T : K \to K$ be a Lipschitzian strongly pseudocontractive mapping. If the fixed point set of T, $F(T)$, is nonempty, then the Mann iteration $\{x_n\} \subset K$ generated by (1.6) with $x_1 \in K$ and the sequence $\{\alpha_n\} \subset (0, 1]$, with $\{\alpha_n\}$ satisfying (1.8) and*

$$\text{(i)} \qquad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

converges strongly to $q \in F(T)$ and $F(T)$ is a single set.

Sastry and Babu [26] noticed that the boundedness of K was not actually used in proving Theorem 1 and hence, exploiting this, they proved a more general result (Theorem 2.2 in the present paper) which, in addition, considered Mann iterations without condition (1.8).

Theorem 2.2. *Let X be a Banach space and K a nonempty closed convex subset of X. If $T : K \to K$ is a Lipschitzian (with constant L) and strongly pseudo-contractive operator (with constant k) such that the fixed point set of T, $F(T)$, is nonempty, then the Mann iteration $\{x_n\} \subset K$ generated by (1.6) with $x_1 \in K$ and the sequence $\{\alpha_n\} \subset (0, 1]$, with $\{\alpha_n\}$ satisfying (i) and*

$$\alpha_n \leq \frac{k - \eta}{(L + 1)(L + 2 - k)}\,,$$

for some $\eta \in (0, k)$, converges strongly to the unique fixed point p of T.

Moreover, there exists $\{\beta_n\}_{n\geq 0}$, a sequence in $(0, 1)$ with $\beta_n \geq (\eta/(1 + k))\alpha_n$, such that for all $n \in \mathbb{N}$, the following estimate holds

$$\|x_{n+1} - p\| \leq \prod_{j=1}^{n} (1 - \beta_j)\, \|x_1 - p\|\,.$$

3. The fastest Krasnoselskij iteration

Due to the fact that condition (1.8) is not assumed, by Theorem 2.2 we can obtain, in particular, a convergence theorem for Krasnoselkij iteration in the class of Lipschitzian and strongly pseudo-contractive operators. This result can, however, be proved independently.

Corollary 3.1. *Let X, K, T, L, k, p be as in Theorem 2.2. Then the Krasnoselskij iteration $\{x_n\} \subset K$ generated by $x_1 \in K$ and (1.4), with $\lambda \in (0, a)$, where*

$$a = \frac{k}{(L+1)(L+2-k)},$$

converges strongly to the (unique) fixed point p of T. Moreover, the following estimate holds

$$\|x_{n+1} - p\| \le q^n \|x_1 - p\|,$$

where

$$q = \frac{1 + (1-k)\lambda + (L+1)(L+2-k)\lambda^2}{1+\lambda}.$$

Proof. Take $\alpha_n \equiv \lambda$ in Theorem 2.2. □

By the previous Corollary we practically obtain a family $\{x_n^\lambda\}$, $\lambda \in (0, a)$ of Krasnoselskij iterative processes such that each of them could be used to approximate the fixed point p of T.

A natural question then arises: which Krasnoselskij iteration from the above family, i.e., which λ, would be more suitable to be considered in order to obtain the better method, if any ?

The answer is given by Theorem 3. To state it, we need the following concept of rate of convergence also used in [1]-[5].

Remind that, in order to compare two fixed point iteration procedures $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ that converge to a certain fixed point p of a given operator T, Rhoades [23] considered that $\{u_n\}$ is *better* than $\{v_n\}$ if

$$\|u_n - p\| \le \|v_n - p\|, \quad \text{for all } n.$$

The terminology from our papers [1]-[5], which is slightly different from that of Rhoades, is more suitable for our purposes here.

Definition 3.1. Let $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$ be two sequences of real numbers that converge to a and b, respectively, and assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

a) If $l = 0$, then we say that $\{a_n\}_{n=0}^\infty$ converges *faster* to a than $\{b_n\}_{n=0}^\infty$ to b;

b) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ *have the same rate of convergence.*

Remarks.

1) In the case a) we use the notation $a_n - a = o(b_n - b)$;

2) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^\infty$ converges faster than $\{a_n\}_{n=0}^\infty$, that is

$$b_n - b = o(a_n - a).$$

Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$, both converging to the same fixed point p, the error estimates

$$\|u_n - p\| \le a_n, \quad n = 0, 1, 2, \ldots \tag{3.11}$$

and

$$\|v_n - p\| \le b_n, \quad n = 0, 1, 2, \ldots \tag{3.12}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero).
Then, in view of Definition 3.1, we introduce the following concept.

Definition 3.2. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (11) and (12), respectively. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then we shall say that $\{u_n\}_{n=0}^{\infty}$ *converges faster* than $\{v_n\}_{n=0}^{\infty}$ to p.

Example 3.2. If we take $p = 0$, $u_n = \dfrac{1}{n+1}$, $v_n = \dfrac{1}{n}$, $n \geq 1$, then $\{u_n\}$ is better than $\{v_n\}$, in the sense of Rhoades [23], but $\{u_n\}$ does not converge faster than $\{v_n\}$. Indeed, we have

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1,$$

and hence in view of Definition 2, $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence. This shows that our Definition 3.2 introduces a sharper concept of rate of convergence than the one considered by Rhoades [23].

The main result of this paper is given by Theorem 3.3.

Theorem 3.3. *Let X be a Banach space and K a nonempty closed convex subset of X. If $T : K \to K$ is a Lipschitzian (with constant L) and strongly pseudo-contractive operator (with constant k) such that the fixed point set of T, $F(T)$, is nonempty, then the Krasnoselskij iteration $\{x_n\} \subset K$ generated by $x_1 \in K$ and (1.4), with $\lambda \in (0, a)$, where*

$$a = \frac{k}{(L+1)(L+2-k)}, \tag{3.13}$$

converges strongly to the (unique) fixed point p of T.
Moreover, among all above Krasnoselskij iterations, there exists one which is the fastest one. It is obtained for

$$\lambda_0 = -1 + \sqrt{1+a},$$

where a is given by (3.13).

Proof. The first part of the proof follows by Theorem 2.2. The second part is elementary: we have to find λ for which the function

$$q(\lambda) = \frac{1 + (1-k)\lambda + (L+1)(L+2-k)\lambda^2}{1+\lambda}$$

attains its minimum value when $\lambda \in (0, a)$, if any. Since $q'(\lambda) = 0$ is equivalent to $\lambda^2 + 2\lambda - a = 0$, we find that $\lambda_0 = -1 + \sqrt{1+a} \in (0, a)$ is the required value of λ. Then, for any $\lambda \in (0, a)$, $\lambda \neq \lambda_0$, we have $\dfrac{q(\lambda_0)}{q(\lambda)} < 1$ and hence $\lim\limits_{n\to\infty} \left(\dfrac{q(\lambda_0)}{q(\lambda)}\right)^n = 0$, which in view of Definition 3.2, shows that $\{x_n^{\lambda_0}\}$ converges faster than $\{x_n^{\lambda}\}$ to the unique fixed point of T. □

4. CONCLUSIONS AND OPEN PROBLEMS

Theorem 3.3 shows that, to efficiently approximate fixed points of Lipschitzian and strictly pseudo-contractive operators, one should always use Krasnoselskij iteration (1.4) and, more specifically, the one obtained for $\lambda_0 = -1 + \sqrt{1+a}$.

It is a current tendency in this field of research to consider more and more complicated fixed point iteration procedures: Ishikawa iteration, Ishikawa iteration with errors, modified Ishikawa iteration etc., see [1], even in the cases when it is known that simpler iteration procedures, like Krasnoselkij's or Mann's are convergent to the fixed point(s) of T. Except for some isolated cases, like Lipschitzian pseudo-contractive operators, see [16], when it is indeed necessary to consider Ishikawa iteration (see the example of Chidume and Mutangadura [13] and also [11], for more details), the use of these complicated iterations is not motivated from a numerical point of view and is not suitable for concrete applications. At most a weak theoretical interest could motivate the numerous papers devoted to this direction of research that appeared in the last decade.

Concluding our note, at least three problems arise:

1. Give an example of operator T, if any, for which some Mann iteration converges and no Krasnoselkij iteration converges to the fixed point(s) of T;

2. Try to transpose known convergence results for Mann iteration based on condition (1.8), to Krasnoselskij iteration, whatever possible;

3. There exists recent papers, we quote here [24], [25], which prove that, for several classes of mappings, Mann iteration is actually equivalent to the more complicated Ishikawa iteration, in the sense that, under certain circumstances, Mann iteration converges (to the fixed point) if and only if Ishikawa converges as well. The challenging problem is then: are Krasnoselskij iteration and Mann iteration equivalent in this sense, for enough large classes of mappings?

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